

Introducing Cadabra: a symbolic computer algebra system for field theory problems

Kasper Peeters

Department of Mathematical Sciences
Durham University
South Road
Durham DH1 3LE
United Kingdom

`kasper.peeters@durham.ac.uk`

Abstract:

Cadabra is a new computer algebra system designed specifically for the solution of problems encountered in field theory. It has extensive functionality for tensor polynomial simplification taking care of Bianchi and Schouten identities, for fermions and anti-commuting variables, Clifford algebras and Fierz transformations, implicit coordinate dependence, multiple index types and many other field theory related concepts. The input format is a subset of \TeX and thus easy to learn. Both a command-line and a graphical interface are available. The present paper is an introduction to the program using several concrete problems from gravity, supergravity and quantum field theory.

This paper describes the capabilities of the Cadabra computer algebra system; the software itself, including source code, can be obtained from <http://www.aei.mpg.de/~peekas/cadabra/> where further documentation can also be found.

Contents

1	Introduction	2
2	Bosonic basics	4
2.1	Tensors, indices and symmetries	4
2.2	Derivatives and dependencies	5
2.3	Bianchi, Ricci and Schouten identities	6
2.4	A Riemann tensor polynomial example	8
2.5	Index ranges and subspaces, Kaluza-Klein gravity	10
3	Fermions, Dirac algebra and Fierz transformations	12
3.1	Simple gamma matrix algebra	12
3.2	Fierz transformations	14
3.3	Other assorted topics	16
4	Conclusions	18

1 Introduction

The currently available spectrum of symbolic computer algebra software exhibits, from the perspective of a high-energy physicist, an unfortunate deficiency. On the one hand, problems dealing with differential equations, symbolic matrix algebra, special functions, series expansions, polynomial algebra and so on are adequately covered, often by several different systems. It is not too hard to translate a give physics problem in these classes from paper to computer and back. Other common problems, on the other hand, such as applying symmetry transformations to Lagrangians, computing Poisson brackets, deriving field equations, canonicalising tensor expressions or performing Fierz transformations, are much harder to handle using existing symbolic computer algebra tools. For these classes of problems, which one could label as “field theory problems”, it typically takes much more effort to convert them from paper and solve them on the computer.

Granted, there are many smaller systems, often built on top of general purpose systems such as Mathematica or Maple, which solve one or more of these problems.¹ However, writing such packages, or even adapting them for slightly more complicated situations than for which they were intended, is often far from easy, and requires a substantial knowledge of the underlying computer algebra system. Moreover, combining them, such that e.g. a tensor manipulating package can deal with fermions too, is typically a tedious exercise, if possible at all. There are clear technical reasons for this, rooted in the design of general purpose computer algebra systems. However, rather than dwelling on these deficiencies, it is more productive to actually come up with a better solution.

The present paper is an introduction to a new computer algebra system, called “Cadabra”, which is designed from the start with field theory problems in mind. By discussing a number

¹Among the more often-used recently written packages there is `GAMMA` for gamma matrix algebra [1], `grassmann.m` for handling of anti-commuting variables [2] and the extensive `xPerm` for tensor manipulation [3]. This list is far from complete and only meant to illustrate how much work one has to do in order to teach a general-purpose symbolic computer algebra system about even the simplest concepts which occur in field theory.

of explicit sample problems representative of the “field theory” class vaguely defined above, this paper is intended as a guide for users rather than computer scientists (for those interested, a description of the technical implementation and code design goals has appeared in [4]).

In which sense does Cadabra differ from other computer algebra systems? The first and most easily visible feature is that all expression input is in the form of a subset of $\text{T}_{\text{E}}\text{X}$. Tensor indices, Dirac conjugation, derivative operators, commutators, fermion products and so on are all written just as in $\text{T}_{\text{E}}\text{X}$. With a little bit of discipline, one can cut-and-paste expressions straight from a paper into a Cadabra notebook. The output, similarly, is typeset as $\text{T}_{\text{E}}\text{X}$ would do it, and Cadabra notebook files are in fact at the same time also valid $\text{T}_{\text{E}}\text{X}$ files (the program comes with a graphical notebook interface, but can also be used from the command line).

Secondly, Cadabra contains some of the most powerful tensor expression simplification routines available at present. Not only does it use simple symmetry or anti-symmetry of tensors or tensor indices in order to simplify expressions², it also takes into account multi-term relations such as the Bianchi identity, as well as dimension-dependent relations such as the Schouten identity (section 2.3). The program handles commuting as well as anti-commuting tensors, allows for multiple index sets (section 2.5), and knows about the concept of a dummy index so that no special wildcard notation is ever needed when handling tensor indices.

Thirdly, the program knows about many concepts which are common in field theory. It handles anti-commuting and non-commuting objects without special notations for their products (section 3.3), it knows about gamma matrix algebra (section 3.1), Fierz identities (section 3.2), Dirac conjugation, vielbeine, flat and curved, covariant and contravariant indices (section 3.3), implicit dependence of tensors on coordinates, partial and covariant derivatives (section 2.2). It has extensive facilities for handling of field theory expressions, e.g. dealing with variational derivatives. It features a substitution command which correctly handles anti-commuting objects and dummy indices and offers a wide variety of pattern matching situations which occur in field theory.

Fourthly, the program has special support that allows the user to write (and keep) expressions in any desired form, e.g. to specify the preferred order of objects. The program does not try to do anything smart unless it is explicitly told to do so. However, when desired, it is always possible to add any arbitrary simplification step to a list of commands which is executed at every step of the calculation. In this way one has precise control over the level of verbosity of the intermediate results of a computation.

Finally, the source code of the program is freely available, with extensive documentation on how to extend it with new algorithms and new data types (e.g. twistor variables are planned for an upcoming release). The program is completely independent of commercial software and relies only on a few other open source libraries and programs. Versions for Linux, Mac OS X and Windows can be downloaded from the web site.

The following sections discuss these characteristic features of Cadabra, illustrated with many explicit calculations and several longer examples (see in particular the Riemann tensor polynomial problem in section 2.4, the Kaluza-Klein gravity problem in section 2.5 and the fermion Fierz problem in section 3.2). These examples were chosen to illustrate the typical use of Cadabra, in particular the way in which it fills a gap in the existing spectrum of computer algebra software. The web site offers a growing collection of longer sample calculations for those readers who are interested in applications to more advanced problems. The program is under active development and information about updates and new features can also be obtained from there.

²I am grateful to José Martín-García for allowing me to use his excellent `xPerm` code [3] for this purpose.

2 Bosonic basics

2.1 Tensors, indices and symmetries

Before discussing actual calculations, let us start with a few words concerning notation. This discussion can be short because, as mentioned in the introduction, mathematical expressions are entered essentially as one would enter them in \TeX (with a few restrictions to avoid ambiguities, which we will discuss as we go along). In order to manipulate expressions, Cadabra often needs to know a bit more about properties of tensors or other symbols. Such properties are entered using the ‘ $::$ ’ symbol. A simple example is the declaration of index sets, useful for automatic dummy index relabelling. An example will clarify this,³

```
1 { a, b, c, d }::Indices.  
2 A_{a b} B_{b c};  
  
A_{ab} B_{bc};
```

(1)

```
3 @substitute!(%)( B_{a b} -> C_{a b c} D_{c} );  
  
A_{ab} C_{bcd} D_d;
```

(2)

The automatic index relabelling which has taken place in this substitute command was clearly only possible because of the property declaration in the first line. Note how the substitute command has also figured out that $B_{\{a b\}}$ on the left-hand side is equivalent to $B_{\{b c\}}$, without any explicit wildcards or patterns. We will see more of this type of field-theory motivated logic throughout the paper.

Indices can be simple letters, as in the example above, but it is also perfectly possible to put accents on them. This can be useful for e.g. notations involving chiral spinors. The following example illustrates the use of accents on indices.

```
1 A_{\dot{a} \dot{b}}::AntiSymmetric.  
2 A_{\dot{b} \dot{a}};  
  
A_{i\dot{a}};
```

(3)

```
3 @canonicalise!(%);  
  
(-1) A_{\dot{a}i};
```

(4)

Here we also see a second usage of property declarations: the construction in the first line declares that the $A_{\dot{a}i}$ tensor is antisymmetric in its indices. The canonicalise command subsequently writes the input in a canonical form, which in this trivial example simply means that the indices gets sorted in alphabetical order. Continuing the example above, one can also use subscripts or superscripts on indices, as in the example below.

³The input and output shown in this paper are essentially identical to those in the graphical interface which comes with Cadabra, the only difference being the added line numbers and some minor typographical modifications.

```

1 { a_{1}, a_{2}, a_{3}, a_{4} }::Indices(vector).
2 V_{a_{1}} W_{a_{1}}:
3 @substitute!(%)( V_{a_{2}} -> M_{a_{2} a_{1}} N_{a_{1}} );

```

$$M_{a_1 a_2} N_{a_2} W_{a_1}; \tag{5}$$

As is clear from these examples, commands always start with a ‘@’ symbol. The ‘!(%)’ bit attached to the commands roughly means that the command should be applied at all levels (the ‘!’ mark) on the previous expression (the ‘%’ sign). Input lines normally end with ‘;’, but output can be suppressed by using ‘.’ instead (as in the last example).

A guiding principle in Cadabra is that nothing ever has to be declared unless this is absolutely needed. This is in contrast to many other systems, where for instance one has to declare manifolds and index sets and so on before one can even enter a tensor expression. This makes code hard to read, but more importantly, such additional declarations are hard to remember. As an example of how Cadabra works, one can e.g. input tensor expressions and perform substitution on them, without ever declaring the symbols used for indices. Only when the program needs to generate new dummy indices does one need to declare index sets, so that dummy indices can be taken out of the right set. The general guideline is that “one only needs to tell the program about the meaning of symbols when this is actually required to do the manipulation correctly”.

In order to declare symmetries of tensors, it is possible to use simple shorthands like the `AntiSymmetric` in the example above. More generally, symmetries can be declared using a generic Young tableau notation. An object with the symmetries of a Riemann tensor, for example, can be declared as in the following example.

```

1 R_{a b c d}::TableauSymmetry(shape={2,2}, indices={0,2,1,3}).
2 R_{a b c d} R_{d c a b}:
3 @canonicalise!(%);

```

$$(-1) R_{abcd} R_{abcd}; \tag{6}$$

The first line indicates that the tensor has the symmetries of the

a	c
b	d

 tableau (the numbers in the `indices` argument refer to the index positions). The `@canonicalise` command writes the input in canonical form with respect to mono-term symmetries (anti-symmetry in the two index pairs and symmetry under pair exchange). The Ricci symmetry is also encoded in the Young tableau and will be discussed later. Many tensor symmetries have shorthand notations, so one often does not have spell out the tableau (e.g. `RiemannTensor` or `SatisfiesBianchi`).

2.2 Derivatives and dependencies

There are relatively few surprises when it comes to derivatives. It is possible to write derivatives with respect to coordinates, i.e. write ∂_x where x is a coordinate, but it is also possible to use indices, as in ∂_i with i being a vector index. It is also possible to make objects implicitly dependent on a derivative operator, so that one can write ∂A without an explicit specification of the coordinate which is involved here.

In order to make this possible, however, derivative objects have to be declared just like indices, otherwise the system does not know which symbol (∂ , D , d , ∇ , ...) one wants to use for them. Implicit dependencies of objects on coordinates associated to derivatives is indicated through a `Depends` property. Here is an example to illustrate all this:

```

1  \nabla{#}::Derivative.
2  \partial{#}::PartialDerivative.
3  A_{m n}::AntiSymmetric.
4  V_{m}::Depends(\nabla).
5
6  \partial_{m p}( A_{q r} V_{n} ) A^{p m};

```

$$\partial_{mp}(A_{qr}V_n)A^{pm}; \tag{7}$$

```

7  @canonicalise!(%);

```

$$0; \tag{8}$$

```

8  \nabla_{m p}( A_{q r} V_{n} ) A^{p m};
9  @canonicalise!(%);

```

$$(-1)\nabla_{mp}(A_{qr}V_n)A^{mp}; \tag{9}$$

```

10 @unwrap!(%);

```

$$(-1)A_{qr}\nabla_{mp}V_nA^{mp}; \tag{10}$$

Note how the symmetry of a double partial derivative has automatically been taken into account (it is part of the `PartialDerivative` property). This is called “property inheritance”.

2.3 Bianchi, Ricci and Schouten identities

So far we have seen several examples of so-called “mono-term” canonicalisation, in which simple symmetries of tensors are used, which relate one particular term to one particular other term. More complicated symmetries are Ricci or Bianchi identities, which relate more than two terms (“multi-term” symmetries). Such identities can be taken into account using Young tableau projectors. Here is an example to show how this works.

```

1  {m,n,p,q,r,s,t#}::Indices(vector).
2  \nabla{#}::Derivative.
3  R_{m n p q}::RiemannTensor.
4  \nabla_{m}{R_{p q r s}}::SatisfiesBianchi.

```

The last line is a shorthand, but internally does nothing more than to associate a particular Young tableau symmetry to the given tensor.⁴ Here it effectively associates the ∇ operator to the Riemann tensor. We can see this in action, for instance, by verifying that the Bianchi identity indeed holds,⁵

```

5 \nabla_{m}{R_{p q r s}} + \nabla_{p}{R_{q m r s}} + \nabla_{q}{R_{m p r s}}:
6 @young_project_tensor!2(%){ModuloMonoterm}:
7 @collect_terms!(%);

0

```

(11)

As expected, by Young-projecting the expression, the Bianchi identity becomes manifest [5].

A similar logic can also be used [6] to take into account dimension-dependent identities, more generally known as Schouten identities or Lovelock identities. Take as an example two anti-symmetric tensors A_{mnp} and B_{mnp} . Then we have

$$A_{mnp}B_{mnq} - A_{mnq}B_{mnp} = 0 \quad \text{in } d = 3, \tag{12}$$

which is conventionally proved by using the fact that anti-symmetrising in four indices yields zero in three dimensions. Cadabra instead uses Young tableau product rules to accomplish the same result [6].

```

1 { m, n, p, q }::Indices(vector).
2 { A_{m n p}, B_{m n p} }::AntiSymmetric.
3 A_{m n p} B_{m n q} - A_{m n q} B_{m n p};

A_{mnp}B_{mnq} - A_{mnq}B_{mnp};

```

(13)

In four dimensions or higher, this product cannot be simplified any further. Decomposing the product using Young projectors and canonicalising with respect to monoterm symmetries gives back the input,

```

4 { m, n, p, q }::Integer(1..4).
5 @decompose_product!(%):
6 @canonicalise!(%):
7 @collect_terms!(%);

A_{pmn}B_{qmn} - A_{qmn}B_{pmn};

```

(14)

In three dimensions, however, the expression vanishes by virtue of the Schouten identity. The same procedure as above now indeed produces a zero,

⁴The alternative is `\nabla_{m}{R_{p q r s}}::TableauSymmetry(shape={3,2}, indices={1,3,0,2,4})`. which makes use of the more general tableau symmetry property which we have seen earlier.

⁵The “2” following the exclamation mark on `@young_project_tensor!2(%)` indicates that this command should only be applied at “level 2” of the expression, i.e. at the level of the terms of the sum.

```

8 { m, n, p, q }::Integer(1..3).
9 @decompose_product!(%):
10 @canonicalise!(%):
11 @collect_terms!(%);

0;

```

Some more examples can be found in [4] and in the reference manual.

As an added bonus, the Young tableau routines which are used internally can also be used directly. There are properties both for filled and unfilled tableaux. The following example shows an example of the latter, together with the Littlewood-Richardson algorithm and the use of the hook formula to compute dimensions of representations,

```

1 \tableau{#}::FilledTableau(dimension=10).
2 \tableau{0,0}{1,1} \tableau{a,a}{b,b}:
3 @lr_tensor!(%);

```

$$\begin{array}{|c|c|c|c|} \hline 0 & 0 & a & a \\ \hline 1 & 1 & b & b \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 0 & 0 & a & a \\ \hline 1 & 1 & b & \\ \hline b & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 0 & 0 & a & a \\ \hline 1 & 1 & & \\ \hline b & b & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 0 & 0 & a \\ \hline 1 & 1 & b \\ \hline a & & \\ \hline b & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 0 & 0 & a \\ \hline 1 & 1 & \\ \hline a & b & \\ \hline b & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 1 & 1 \\ \hline a & a \\ \hline b & b \\ \hline \end{array};$$

```

4 @tabdimension!(%);

70785 + 235950 + 154440 + 101640 + 103950 + 13860;

```

2.4 A Riemann tensor polynomial example

Let us now discuss a somewhat more complicated example of bosonic tensor algebra, which shows not only how symmetries of tensors are used, but also how repeated derivative operators work and how more complicated canonicalisation problems are solved.

To this extent we will discuss a certain identity, a proof of which can be found in appendix A of [7]. Concretely, the problem concerns two particular third-order polynomials in the Weyl tensor C_{mnpq} , given by

$$\begin{aligned}
E_{ij} &= -C_i^{mkl} C_{jpkq} C_l^{pmq} + \frac{1}{4} C_i^{mkl} C_{jmpq} C_{kl}^{pq} - \frac{1}{2} C_{ikjl} C^{kmpq} C^l_{mpq}, \\
E &= C_{jmnk} C^{mpqn} C_p^{jk}_q + \frac{1}{2} C_{jkmn} C^{pqmn} C^{jk}_{pq}.
\end{aligned}$$

When evaluated on an Einstein space, these polynomials are supposed to satisfy the identity

$$\nabla_i \nabla_j E_{ij} - \frac{1}{6} \nabla_i \nabla_i E = 0.$$

Proving this is a tedious exercise with Bianchi identities when done by hand. With Cadabra the proof is straightforward; the notebook is depicted below.

The key elements in this example are the declaration of the Weyl tensor as well as the covariant derivative, together with the 4th line which links the two, by stating that the covariant derivative of the Weyl tensor satisfies the Bianchi identity.

```

1 {i,j,m,n,k,p,q,l,r,r#}::Indices(vector).
2 C_{m n p q}::WeylTensor.
3 \nabla_{#}::Derivative.
4 \nabla_{r}{ C_{m n p q} }::SatisfiesBianchi.
5
6 Eij:=- C_{i m k l} C_{j p k q} C_{l p m q} + 1/4 C_{i m k l} C_{j m p q} C_{k l p q}
7       - 1/2 C_{i k j l} C_{k m p q} C_{l m p q}:
8
9 E:= C_{j m n k} C_{m p q n} C_{p j k q} + 1/2 C_{j k m n} C_{p q m n} C_{j k p q}:
10
11 exp:= \nabla_{i}{\nabla_{j}{ @ (Eij) }} - 1/6 \nabla_{i}{\nabla_{i}{ @ (E) }}:

```

We now need to apply (twice) the Leibniz rule to expand the derivatives, and then sort the tensors and write the result in canonical form with respect to mono-term symmetries,

```

12 @distribute! (%): @prodrule! (%):
13 @distribute! (%): @prodrule! (%):
14
15 @prodsort! (%): @canonicalise! (%): @rename_dummies! (%):
16 @collect_terms! (%):

```

Because the identity which we intend to prove is only supposed to hold on Einstein spaces, we set the divergence of the Weyl tensor to zero,

```

17 @substitute! (%)( \nabla_{i}{C_{k i l m}} -> 0 , \nabla_{i}{C_{k m l i}} -> 0 );

```

$$\begin{aligned}
& C_{ijmn}C_{ikmp}\nabla_q\nabla_jC_{nkpq} - C_{ijmn}\nabla_kC_{ipmq}\nabla_pC_{jqnk} - 2C_{ijmn}\nabla_iC_{mkpq}\nabla_pC_{jknq} \\
& - C_{ijmn}C_{ikpq}\nabla_m\nabla_pC_{jqnk} - \frac{1}{4}C_{ijmn}C_{ijkp}\nabla_q\nabla_mC_{nqkp} + \frac{1}{4}C_{ijmn}\nabla_kC_{ijpq}\nabla_pC_{mnkq} \\
& - \frac{1}{2}C_{ijmn}\nabla_iC_{jkpq}\nabla_kC_{mnpq} + \frac{1}{4}C_{ijmn}C_{ikpq}\nabla_j\nabla_kC_{mnpq} + \frac{1}{2}C_{ijmn}C_{ikpq}\nabla_m\nabla_jC_{nkpq} \\
& + \frac{1}{2}C_{ijmn}\nabla_iC_{mkpq}\nabla_nC_{jkpq} - \frac{1}{2}C_{ijmn}\nabla_iC_{jkpq}\nabla_mC_{nkpq} + \frac{1}{2}C_{ijmn}C_{ikpq}\nabla_j\nabla_mC_{nkpq} \\
& + \frac{1}{2}C_{ijmn}C_{ikmp}\nabla_q\nabla_qC_{jknq} + C_{ijmn}\nabla_kC_{ipmq}\nabla_kC_{jpnq} - \frac{1}{4}C_{ijmn}C_{ijkp}\nabla_q\nabla_qC_{mnkq} \\
& - \frac{1}{2}C_{ijmn}\nabla_kC_{ijpq}\nabla_kC_{mnpq};
\end{aligned} \tag{20}$$

This expression should vanish upon use of the Bianchi identity. By expanding all tensors using their Young projectors, this becomes manifest,

```

18 @young_project_product! (%):
19 @sumflatten! (%):
20 @collect_terms! (%);

```

$$0; \tag{21}$$

This proves the identity (19).

The use of Young projector methods also allows for other calculations which are computationally intensive when done by hand. An example is the problem of generating a complete basis of monomials of Riemann tensors, discussed in [4].

2.5 Index ranges and subspaces, Kaluza-Klein gravity

We have so far seen rather simple uses of indices, in which only a single vector space was used. Cadabra contains functionality to deal with more complicated situations though. In section 3 we will discuss the use of multiple index types as well as implicit indices. Here we will discuss another index feature, namely the possibility to “split” indices into two or more subspaces. Let us first illustrate this with a simple example.

```

1 {M, N, P}::Indices(space).
2 {m, n, p}::Indices(subspace1).
3 {a, b, c}::Indices(subspace2).
4
5 A_{M N} B_{N P};
6 @split_index!({M, m, a});

```

$$A_{Mm} B_{mP} + A_{Ma} B_{aP}; \tag{22}$$

The first three lines declare three types of indices, labelled as “space”, “subspace1” and “subspace2” respectively. The last line of the input performs the split of the dummy index into the two subspaces. Instead of using two subspaces labelled by indices, it is also possible to use a one-dimensional subspace. An example of how this works can be found in the Kaluza-Klein problem discussed below.

The index split functionality is particularly useful for Kaluza-Klein type problems, in which the metric is decomposed according to

$$g_{\mu\nu} = \begin{pmatrix} \phi^{-1} h_{mn} + \phi A_m A_n & \phi A_m \\ \phi A_n & \phi \end{pmatrix}, \tag{23}$$

(where the usual conventions in four space-time dimensions were used). It is a somewhat tedious exercise to compute the Riemann tensor components for this particular metric ansatz. One finds for instance that

$$R_{m4n4} = -\frac{1}{2}\nabla_m\partial_n\phi - \frac{1}{4}\partial_m\phi\partial_n\phi\phi^{-1} + \frac{1}{4}\partial_p\phi\partial_q\phi\phi^{-1}h_{mn}h^{pq} + \frac{1}{4}F_{mp}F_{nq}\phi^3h^{pq}, \tag{24}$$

where F_{mn} is the field strength of A_m . The following notebook computes this expression using Cadabra and the index splitting functionality. All intermediate output is suppressed as it tends to get rather lengthy.

```

1 {\mu,\nu,\rho,\sigma,\kappa,\lambda,\eta,\chi#}::Indices(full, position=fixed).
2 {m,n,p,q,r,s,t,u,v,m#}::Indices(subspace, position=fixed, parent=full).

```

Note the appearance of `parent=full`. This indicates that the indices in the second set span a subspace of the indices in the first set. The remaining declarations are standard,

```

3 \partial{#}::PartialDerivative.
4 g_{\mu\nu}::Metric.
5 g^{\mu\nu}::InverseMetric.
6 g_{\mu? \nu?}::Symmetric.
7 g^{\mu? \nu?}::Symmetric.

```

```

8 h_{m n}::Metric.
9 h^{m n}::InverseMetric.
10 \delta^{\mu?}_{\nu?}::KroneckerDelta.
11 \delta_{\mu?}^{\nu?}::KroneckerDelta.
12 F_{m n}::AntiSymmetric.

```

We will want to expand the Riemann tensor in terms of the metric. The following two substitution rules do the conversion from Riemann tensor to Christoffel symbol and from Christoffel symbol to metric.⁶ Index patterns like `\lambda?` match both four- and three-dimensional indices.

```

13 RtoG:= R^{\lambda?}_{\mu?\nu?\kappa?} ->
14   - \partial_{\kappa?}{ \Gamma^{\lambda?}_{\mu?\nu?} }
15   + \partial_{\nu?}{ \Gamma^{\lambda?}_{\mu?\kappa?} }
16   - \Gamma^{\eta?}_{\mu?\nu?} \Gamma^{\lambda?}_{\kappa?\eta?}
17   + \Gamma^{\eta?}_{\mu?\kappa?} \Gamma^{\lambda?}_{\nu?\eta?}:
18
19 Gtog:= \Gamma^{\lambda?}_{\mu?\nu?} ->
20   (1/2) * g^{\lambda?\kappa?} (
21     \partial_{\nu?}{ g_{\kappa\mu?} } + \partial_{\mu?}{ g_{\kappa\nu?} }
22     - \partial_{\kappa?}{ g_{\mu?\nu?} } ):

```

Now input the R_{m4n4} component and do the substitution. After each substitution, we distribute products over sums. We also apply the product rule to distribute derivatives over factors in a product.

```

23 todo:= g_{m1 m} R^{m1}_{4 n 4} + g_{4 m} R^{4}_{4 n 4};
24 @substitute!(%)( @RtoG );
25 @substitute!(%)( @Gtog );
26 @distribute!(%);
27 @prodrule!(%);
28 @distribute!(%);
29 @prodsort!(%);

```

We now split the μ index into a m part and the remaining 4 direction (the `!!` version of the command makes it apply until the result no longer changes). After that, we remove x^4 derivatives of the gauge field and write the expression in canonical form,

```

30 @split_index!!(%){\mu,m1,4};
31 @canonicalise!(%):
32 @substitute!(%)( \partial_{4}{A??} -> 0 );
33 @substitute!(%)( \partial_{4 m?}{A??} -> 0 );

```

In the next step, we insert the metric ansatz (23) and simplify the result as much as possible.

```

34 @substitute!(%)( g_{4 4} -> \phi );
35 @substitute!(%)( g_{4 m} -> \phi A_{m} );
36 @substitute!(%)( g_{m n} -> \phi**{-1} h_{m n} + \phi A_{m} A_{n} );
37 @substitute!(%)( g^{4 4} -> \phi**{-1} + \phi A_{m} h^{m n} A_{n} );
38 @substitute!(%)( g^{4 m} -> - \phi h^{m n} A_{n} );

```

⁶A library with common substitution rules like these is in preparation.

```

39 @substitute!(%)( g^{m n} -> \phi h^{m n} ):
40 @distribute!(%):
41 @prodrule!(%):
42 @distribute!(%):
43 @prodrule!(%);
44 @distribute!(%);
45 @canonicalise!(%):
46 @substitute!!(%)( h_{m1 m2} h^{m3 m2} -> \delta_{m1}^{m3} );
47 @eliminate_kr!(%):

```

Some derivatives have to be rewritten to a canonical form,

```

48 @substitute!(%)( \partial_{m}\{\phi^{*-1}\} -> -\phi^{*-2} \partial_{m}\{\phi\} ):
49 @collect_factors!(%):
50 @prodsort!(%):
51 @substitute!(%)( \partial_{p}\{h^{n m}\} h_{q m} -> - \partial_{p}\{h_{q m}\} h^{n m} );
52 @canonicalise!(%):
53 @substitute!(%)( h_{m1 m2} h^{m3 m2} -> \delta_{m1}^{m3} );
54 @eliminate_kr!(%):

```

Finally, we replace the derivative of the gauge field with the field strength,

```

55 @substitute!(%)( \partial_{n}\{A_{m}\} -> 1/2*\partial_{n}\{A_{m}\} + 1/2*F_{n m}
56                                     + 1/2*\partial_{m}\{A_{n}\} ):
57 @distribute!(%):
58 @prodsort!(%):
59 @canonicalise!(%):
60 @rename_dummies!(%):
61 @collect_terms!(%):

```

$$\begin{aligned}
& -\frac{1}{4}\partial_m\phi\partial_n\phi\phi^{-1} + \frac{1}{4}\partial_p\phi\partial_n h_{mq}h^{pq} - \frac{1}{2}\partial_{mn}\phi + \frac{1}{4}F_{mp}F_{nq}\phi^3h^{pq} \\
& + \frac{1}{4}\partial_p\phi\partial_q\phi\phi^{-1}h_{mn}h^{pq} - \frac{1}{4}\partial_p\phi\partial_q h_{mn}h^{pq} + \frac{1}{4}\partial_p\phi\partial_m h_{nq}h^{pq};
\end{aligned} \tag{25}$$

This is indeed equivalent to (24) upon writing out the covariant derivative in the first term of that equation.

If required, some of these calculations can be done with fewer lines of input by adding a number of default simplification rules; an example of such default rules will be discussed in the next section. We will end here the discussion of purely bosonic problems and move on to fermions and anti-commuting tensors.

3 Fermions, Dirac algebra and Fierz transformations

3.1 Simple gamma matrix algebra

Cadabra has built-in algorithms for the manipulation of anti-commuting objects, spinors and gamma matrices in any dimension. Combined with the option of “suppressing” indices (in our examples below we will suppress spinor indices), it becomes possible to write calculations in a

natural and compact way. All anti-commuting and fermionic objects as usual need to be declared by attaching the appropriate properties to them; we will see many examples of this.

Let us start, however, with some simple gamma matrix algebra. As an example, we will expand the product $\Gamma_{sr}\Gamma_{rl}\Gamma_{km}\Gamma_{ms}$ in arbitrary dimensions in terms of the irreducible Γ_{kl} and δ_{kl} components.

```
1  ::PostDefaultRules( @@prodsort!(%), @@eliminate_kr!(%),
2                          @@canonicalise!(%), @@collect_terms!(%) ).
```

This single line adds some default simplification rules to the system, so that we do not have to type these by hand at every stage of the calculation.

We now declare the vector indices, their range, and the symbols used for gamma matrices and Kronecker deltas.

```
3  {s,r,l,k,m,n}::Indices(vector).
4  {s,r,l,k,m,n}::Integer(0..d-1).
5  \Gamma_{#}::GammaMatrix(metric=\delta).
6  \delta_{m n}::KroneckerDelta.
```

The declaration for the gamma matrix shows that we are defining an object with implicit indices: the spinor indices will be suppressed. The notation $_{\#}$ denotes the presence of an arbitrary number of indices.

Next follows the actual calculation.

```
7  \Gamma_{s r} \Gamma_{r l} \Gamma_{k m} \Gamma_{m s}:
8  @join!(%){expand}:
9  @join!(%){expand};
```

$$(-1)((2\Gamma_{lr} - \Gamma_{lr}d + \delta_{lr}d - \delta_{lr})(2\Gamma_{kr} - \Gamma_{kr}d + \delta_{kr} - \delta_{kr}d)); \quad (26)$$

```
10 @distribute!(%);
11 @join!(%){expand};
12 @distribute!(%);
13 @factorise!(%){d};
14 @collect_factors!(%);
```

$$\Gamma_{kl}(12 - 18d + 8d^2 - d^3) + \delta_{kl}(-3 + 6d - 4d^2 + d^3); \quad (27)$$

The key ingredient here is the `@join` command, which takes two adjacent gamma matrices and expands their product in terms of a basis of fully antisymmetrised gamma matrices. In the step before (26) the two commands join the 1st and 2nd gamma matrix, as well as the 3rd and 4th. The product is then expanded out and one final gamma join operation is performed. The last two lines combine all the d -dependence in overall factors.

In the example above, the spinor indices on the gamma matrices were suppressed. The `GammaMatrix` property has turned the gamma symbols into non-commuting objects, which will not change order when sorting symbols in a product. It is, however, possible to add the spinor indices back in, and use explicit indices. For this purpose, Cadabra knows the concept of an “index bracket”, which associates indices to matrix objects like the gamma matrices above, or to e.g. vectors or spinors. Here is a somewhat simpler example:

```

1 {a,b,c,d#}::Indices(spinor).
2 \Gamma_{#}::GammaMatrix(metric=\delta).
3 (\Gamma_{m n})_{a b} (\Gamma_{n p})_{b c};
4 @combine!(%);

```

$$(\Gamma_{mn} \Gamma_{np})_{ac}; \tag{28}$$

```

5 @join!(%){expand}:
6 @canonicalise!(%);

```

$$(\Gamma_{mp} \delta_{nn} - \Gamma_{mn} \delta_{np} + \Gamma_{pn} \delta_{mn} + \delta_{mp} \delta_{nn} - \delta_{mn} \delta_{np})_{ac}; \tag{29}$$

The join algorithm has acted ‘inside’ the index bracket, on the matrix objects themselves. Conversely, index brackets can be distributed over elements in the sum, and objects can be taken out of the index bracket when they are known not to contain implicit indices:

```

7 @distribute!(%):
8 @expand!(%);

```

$$(\Gamma_{mp})_{ac} \delta_{nn} - (\Gamma_{mn})_{ac} \delta_{np} + (\Gamma_{pn})_{ac} \delta_{mn} + (\delta_{mp} \delta_{nn})_{ac} - (\delta_{mn} \delta_{np})_{ac}; \tag{30}$$

In this way, component and abstract calculations can be mixed at will.

3.2 Fierz transformations

Cadabra can apply Fierz transformations in any dimension to re-order four-fermion terms. As an example, consider the following identity for Majorana spinors in eleven dimensions [8],

$$\begin{aligned}
-e_{[\nu}{}^s (\bar{\theta} \Gamma^{rs} \psi_{\rho}) (\bar{\psi}_{\mu]} \Gamma^r \epsilon) &= \sum_n \frac{1}{2^5 n!} (\bar{\psi}_{\mu} \Gamma^{i_1 \dots i_n} \psi_{\rho}) (\bar{\theta} \Gamma^{rs} \Gamma^{i_n \dots i_1} \Gamma^r \epsilon) \\
&= \frac{1}{2^5} \bar{\psi}_{[\mu} \Gamma_m \psi_{\rho} e_{\nu]}{}^s (8 \bar{\theta} \Gamma^{sm} \epsilon + 10 \eta^{sm} \bar{\theta} \epsilon) \\
&\quad - \frac{1}{2^5 2!} \bar{\psi}_{[\mu} \Gamma_{mn} \psi_{\rho} e_{\nu]}{}^s (-6 \bar{\theta} \Gamma^{smn} \epsilon + 16 \bar{\theta} \Gamma^{[m} \epsilon \eta^{n]s}) \\
&\quad + \frac{1}{2^5 5!} \bar{\psi}_{[\mu} \Gamma_{mnopq} \psi_{\rho} e_{\nu]}{}^s (10 \eta^{s[m} \bar{\theta} \Gamma^{nopq]} \epsilon).
\end{aligned} \tag{31}$$

With conventional tools, it would take some time to convert this problem to the computer. Proving this with Cadabra, on the other hand, requires little more than defining symbols properly and inputting the expression on the left-hand side as one would type it in a paper. Moreover, reading off the output is simple as well, because Cadabra produces output which is virtually identical to the right-hand side of the equation above. The notebook with comments is displayed below.

```

1  {\mu,\nu,\rho}::Indices(curved, position=fixed).
2  {m,n,p,q,r,s,t#}::Indices(flat).
3  {m,n,p,q,r,s,t#}::Integer(0..10).
4  T^{#\mu}::AntiSymmetric.
5  \psi_{\mu}::SelfAntiCommuting.
6  \psi_{\mu}::Spinor(dimension=11, type=Majorana).
7  \theta::Spinor(dimension=11, type=Majorana).
8  \epsilon::Spinor(dimension=11, type=Majorana).
9  {\theta,\epsilon,\psi_{\mu}}::AntiCommuting
10 \bar{#}::DiracBar.
11 \delta^{m n}::KroneckerDelta.
12 \Gamma^{#\{m\}}::GammaMatrix(metric=\delta).

```

These lines define the properties of all the symbols. We now input (minus) the left-hand side of (31), and do a Fierz transformation to bring the θ and ϵ spinors together,

```

13 T^{\mu\nu\rho} e_{\nu}^{\{s}
14     \bar{\theta} \Gamma^{rs} \psi_{\rho}
15     \bar{\psi}_{\mu} \Gamma^r \epsilon;
16
17 @fierz!(%)( \theta, \epsilon, \psi_{\mu}, \psi_{\nu} );

```

$$\begin{aligned}
& -\frac{1}{32} T^{\mu\nu\rho} e_{\nu}^s \bar{\theta} \Gamma^{rs} \Gamma^r \epsilon \bar{\psi}_{\mu} \psi_{\rho} - \frac{1}{32} T^{\mu\nu\rho} e_{\nu}^s \bar{\theta} \Gamma^{rs} \Gamma^m \Gamma^r \epsilon \bar{\psi}_{\mu} \Gamma_m \psi_{\rho} \\
& -\frac{1}{64} T^{\mu\nu\rho} e_{\nu}^s \bar{\theta} \Gamma^{rs} \Gamma^{mn} \Gamma^r \epsilon \bar{\psi}_{\mu} \Gamma_{nm} \psi_{\rho} - \frac{1}{192} T^{\mu\nu\rho} e_{\nu}^s \bar{\theta} \Gamma^{rs} \Gamma^{mnp} \Gamma^r \epsilon \bar{\psi}_{\mu} \Gamma_{pnm} \psi_{\rho} \\
& -\frac{1}{768} T^{\mu\nu\rho} e_{\nu}^s \bar{\theta} \Gamma^{rs} \Gamma^{mnpq} \Gamma^r \epsilon \bar{\psi}_{\mu} \Gamma_{qpnm} \psi_{\rho} - \frac{1}{3840} T^{\mu\nu\rho} e_{\nu}^s \bar{\theta} \Gamma^{rs} \Gamma^{mnpqt1} \Gamma^r \epsilon \bar{\psi}_{\mu} \Gamma_{t1qpnm} \psi_{\rho};
\end{aligned} \tag{32}$$

This is not yet in the desired form, so we join gamma matrices twice and canonicalise the resulting expression,

```

18 @join!(%){expand}:
19 @distribute!(%):
20 @eliminate_kr!(%):
21 @join!(%){expand}:
22 @distribute!(%):
23 @eliminate_kr!(%):
24 @collect_terms!(%):
25 @canonicalise!(%):
26 @rename_dummies!(%):
27 @collect_terms!(%);

```

$$\begin{aligned}
& \frac{1}{4} T^{\mu\nu\rho} e_{\mu}^m \bar{\theta} \Gamma^{mn} \epsilon \bar{\psi}_{\nu} \Gamma_n \psi_{\rho} + \frac{5}{16} T^{\mu\nu\rho} e_{\mu}^m \bar{\theta} \epsilon \bar{\psi}_{\nu} \Gamma_m \psi_{\rho} + \frac{3}{32} T^{\mu\nu\rho} e_{\mu}^m \bar{\theta} \Gamma^{mnp} \epsilon \bar{\psi}_{\nu} \Gamma_{np} \psi_{\rho} \\
& + \frac{1}{4} T^{\mu\nu\rho} e_{\mu}^m \bar{\theta} \Gamma^n \epsilon \bar{\psi}_{\nu} \Gamma_{mn} \psi_{\rho} + \frac{1}{384} T^{\mu\nu\rho} e_{\mu}^m \bar{\theta} \Gamma^{npqr} \epsilon \bar{\psi}_{\nu} \Gamma_{mnpqr} \psi_{\rho};
\end{aligned} \tag{33}$$

This is indeed equivalent to the right-hand side of (31).

Note in particular once more the simplicity of the input on line 13-17. As advertised, very little knowledge of the program is needed in order to be able to read and follow a calculation in a Cadabra notebook.

3.3 Other assorted topics

So far we have only seen the substitution command in action on bosonic objects. However, the substitution command in Cadabra is aware of anti-commuting objects as well, and will take care of signs whenever products in the pattern and the expression are not ordered in the same way. It will also refuse to match a pattern if two symbols are declared non-commuting, and do not appear in the same order in the pattern and in the expression.

Anti-commutativity comes in two flavours: self-anticommutativity and mutual anti-commutativity. The first is used for objects which carry an index: if ψ_μ is declared self-anticommuting, it means that $\psi_\mu\psi_\nu = -\psi_\nu\psi_\mu$. Below is an example to illustrate these concepts and the substitution functionality:

```

1 \psi_{\mu}::SelfAntiCommuting.
2 { \chi, \psi_{\mu} }::AntiCommuting.
3 \chi A^{\mu\nu} \psi_{\mu} \chi \psi_{\nu};

```

$$\chi A^{\mu\nu} \psi_\mu \chi \psi_\nu; \tag{34}$$

```

1 @substitute!(%)( \psi_{\mu} \psi_{\nu} -> B_{\mu\nu} );

```

$$(-1)\chi A^{\mu\nu} B_{\mu\nu} \chi; \tag{35}$$

By declaring χ and ψ_μ to be mutually anti-commuting, Cadabra knows that a sign should be picked up when doing the substitution.

In order to illustrate the meaning of `SelfAntiCommuting`, we undo the substitution command above,

```

1 @pop(%);

```

$$\chi A^{\mu\nu} \psi_\mu \chi \psi_\nu; \tag{36}$$

Now we declare $A^{\mu\nu}$ to be symmetric and canonicalise the expression,

```

1 A^{\mu\nu}::Symmetric.
2 @canonicalise!(%);

```

$$0; \tag{37}$$

Because ψ_μ has been declared `SelfAntiCommuting`, the program knows that $\psi_\mu\psi_\nu$ is anti-symmetric in its two indices, and has used this to simplify the expression.

Incidentally, the `@pop` command provides an undo facility which allows one to go back any arbitrary number of steps in the calculation (by simply applying it multiple times).

To conclude, let us discuss one more calculation which combines some of the functionality of Cadabra discussed above, and also shows how to handle variational problems, especially those involving fermionic objects. An example is the variation of a Lagrangian under supersymmetry transformations. Consider the following sample calculation,

```

1  ::PostDefaultRules( @@eliminate_kr!(%), @@prodsort!(%), @@collect_terms!(%) ).

2  D{#}::Derivative.
3  \bar{#}::DiracBar.
4  \delta{A??}::Derivative.
5  {m,n,p,q,r,s,t,u,m#}::Indices(flat).
6  {m,n,p,q,r,s,t,u,m#}::Integer(0..3).
7  {\mu,\nu,\rho,\sigma,\kappa,\lambda,\alpha,\beta}::Indices(curved,position=fixed).
8  {\mu,\nu,\rho,\sigma,\kappa,\lambda,\alpha,\beta}::Integer(0..3).

```

Declaration of bosonic fields,

```

9  e^{m \mu}::Vielbein.
10 e_{m \mu}::InverseVielbein.
11 g^{\mu\nu}::InverseMetric.
12 g_{\mu\nu}::Metric.

```

Declaration of fermionic fields,

```

13 { \epsilon,\psi_{\mu},\psi_{\mu\nu} }::Spinor(dimension=4, type=Majorana).
14 \Gamma_{#{m}}::GammaMatrix(metric=\delta).
15 { \psi_{\mu\nu}, \psi_{\mu}, \epsilon }::AntiCommuting.
16 { \psi_{\mu}, \psi_{\mu\nu} }::SelfAntiCommuting.
17 { \epsilon, \psi_{\mu}, \psi_{\mu\nu} }::SortOrder.
18 \Gamma_{#{m}}::Depends(\bar).
19 \psi_{\mu\nu}::AntiSymmetric.

```

Input of the Lagrangian and rewriting such that all indices on gamma matrices are flat,

```

20 L:= -1/2 e e^{n \nu} e^{m \mu} R_{\{\mu\nu n m\}
21         - 1/2 e \bar{\psi}_{\mu} \Gamma^{\{\mu\nu\rho\}} D_{\nu}\{\psi_{\rho}\}:
22 @rewrite_indices!({ \Gamma^{m n p} }{ e^{n \mu} });

```

$$L := -\frac{1}{2}R_{\mu\nu n m}e^{m\mu}e^{n\nu} - \frac{1}{2}\bar{\psi}_{\mu}\Gamma^{mnp}D_{\nu}\psi_{\rho}e^{m\mu}e^{n\nu}e^{p\rho}; \quad (38)$$

In the 1.5th order formalism, the supersymmetry transformation rules are given by

```

23 susy:= { e^{n \mu} -> -\bar{\epsilon}\Gamma^m \psi_{\nu} e^{m \mu} e^{n \nu},
24         e         -> e \bar{\epsilon}\Gamma^n \psi_{\mu} e^{n \mu},
25         \psi_{\mu} -> D_{\mu}\{\epsilon\} };

```

Varying under supersymmetry is now a simple matter of using the @vary command, giving it the rules defined above. This will automatically assume infinitesimal variations.

```

26 @vary!(L)( @(susy) );

```

$$\begin{aligned}
L := & -\frac{1}{2} R_{\mu\nu nm} \bar{\epsilon} \Gamma^p \psi_\rho e e^{p\rho} e^{m\mu} e^{n\nu} + \frac{1}{2} R_{\mu\nu nm} e \bar{\epsilon} \Gamma^p \psi_\rho e^{p\mu} e^{m\rho} e^{n\nu} \\
& + \frac{1}{2} R_{\mu\nu nm} e e^{m\mu} \bar{\epsilon} \Gamma^p \psi_\rho e^{p\nu} e^{n\rho} - \frac{1}{2} \Gamma^{mnp} \overline{D_\mu \epsilon} D_\nu \psi_\rho e e^{m\mu} e^{n\nu} e^{p\rho} \\
& - \frac{1}{2} \Gamma^{mnp} \overline{\psi_\mu} D_\nu D_\rho e e^{m\mu} e^{n\nu} e^{p\rho} - \frac{1}{2} \Gamma^{mnp} \overline{\psi_\mu} D_\nu \psi_\rho \bar{\epsilon} \Gamma^q \psi_\sigma e e^{q\sigma} e^{m\mu} e^{n\nu} e^{p\rho} \\
& + \frac{1}{2} \Gamma^{mnp} \overline{\psi_\mu} D_\nu \psi_\rho e \bar{\epsilon} \Gamma^q \psi_\sigma e^{q\mu} e^{m\sigma} e^{n\nu} e^{p\rho} + \frac{1}{2} \Gamma^{mnp} \overline{\psi_\mu} D_\nu \psi_\rho e e^{m\mu} \bar{\epsilon} \Gamma^q \psi_\sigma e^{q\nu} e^{n\sigma} e^{p\rho} \\
& + \frac{1}{2} \Gamma^{mnp} \overline{\psi_\mu} D_\nu \psi_\rho e e^{m\mu} e^{n\nu} \bar{\epsilon} \Gamma^q \psi_\sigma e^{q\rho} e^{p\sigma};
\end{aligned} \tag{39}$$

(The rest of the calculation, in which the result is rewritten using partial integration and a Fierz identity to show that it vanishes, can be found in a more extensive notebook on $N = 1$ supergravity in four dimensions, available from the web site).

Various aspects of Cadabra are visible here: the use of multiple index types and conversion between them, the user-specified sort order of fields which takes into account commutativity properties of tensors, the pattern matching routines which automatically deal with dummy index relabelling, the readability of input/output and so on.

4 Conclusions

We have discussed the capabilities of the new computer algebra system ‘‘Cadabra’’, by applying it to a number of concrete calculations. There are several aspects of this system which make it particularly well-suited to solve field-theory problems. Firstly, it uses (a subset of) \TeX not only for output, but also for input. Compared to other computer algebra systems, this makes it much easier to input complicated expressions without errors, as there is hardly any new syntax to be learnt. Secondly, the program has built-in facilities for many concepts which occur in field theory, like anti-commuting variables, gamma matrix algebra, implicit dependence on coordinates, accents, multiple dummy index sets, canonicalisation of tensor expressions and so on. Although there exist other systems which share some of the functionality of Cadabra, the implementation in Cadabra was designed from scratch so as to make problem solving resemble as close as possible the steps one would follow with pencil and paper.

The program is entirely built on freely distributable software libraries, i.e. it does not make use of any proprietary computer algebra system. The system is available in source code as well as in binary form for Linux and Mac OS X computers; the reader is referred to the web site <http://www.aei.mpg.de/~peekas/cadabra/> for download and installation instructions as well as the address of the mailing list.

Given the program’s scope and size, and the fact that this is only the first public release, there of course remains quite some room for improvements and additions. Planned future enhancements include e.g. two-component spinors, automatic canonicalisation using Fierz identities, and twistor objects.

Acknowledgements

The development of Cadabra was made possible by support from DAMTP at Cambridge University, from CERN and in particular from the Albert-Einstein-Institute in Potsdam. This work was also sponsored in part by VIDI grant 016.069.313 from the Dutch Organisation for Scientific Research (NWO). I am grateful to the Department of Mathematical Sciences at Durham University for hospitality while this work was being completed. Many thanks to Marcus Berg for comments on the first version of this paper.

References

- [1] U. Gran, “GAMMA: A Mathematica package for performing gamma-matrix algebra and Fierz transformations in arbitrary dimensions”, `hep-th/0105086`, <http://fy.chalmers.se/~gran/GAMMA/>.
- [2] M. Headrick, “grassmann.m”, <http://www.stanford.edu/~headrick/physics/index.html>.
- [3] J. Martín-García, “xPerm and xAct”, <http://metric.iem.csic.es/Martin-Garcia/xAct/index.html>.
- [4] K. Peeters, “A field-theory motivated approach to symbolic computer algebra”, *Comp. Phys. Commun.* **176** (2006) 550–558, `cs.sc/0608005`.
- [5] M. B. Green, K. Peeters, and C. Stahn, “Superfield integrals in high dimensions”, *JHEP* **08** (2005) 093, `hep-th/0506161`.
- [6] K. Peeters, “A Young tableau based algorithm for dimensionally dependent identities”, in preparation.
- [7] S. Frolov and A. A. Tseytlin, “ R^4 corrections to conifolds and $G(2)$ holonomy metrics”, *Nucl. Phys.* **B632** (2002) 69–100, `hep-th/0111128`.
- [8] B. de Wit, K. Peeters, and J. Plefka, “Superspace geometry for supermembrane backgrounds”, *Nucl. Phys.* **B532** (1998) 99, `hep-th/9803209`.